

# SVD AND PD BASED PROJECTION METRICS ON SE(N)

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**Abstract** An open research question is how to define a metric on SE(n) that is as invariant as possible with respect to (1) the choice of coordinate frames and (2) the units used to measure linear and angular distances. We present two techniques for approximating elements of the special Euclidean group SE(n) with elements of the special orthogonal group SO(n+1). These techniques are based on the singular value and polar decompositions (denoted as SVD and PD respectively) of the homogeneous transform representation of the elements of SE(n). The projection of the elements of SE(n) onto SO(n+1) yields hyperdimensional rotations that approximate the rigid-body displacements. Any of the infinite bi-invariant metrics on SO(n+1) may then be used to measure the *distance* between any two spatial displacements. The results are PD and SVD based projection techniques that yield two approximately bi-invariant metrics on SE(n). These metrics have applications in motion synthesis, robot calibration, motion interpolation, and hybrid robot control.

**Keywords:** Displacement metrics, metrics on the special Euclidean group, rigid-body displacements

## 1. Introduction

Simply stated a metric measures the distance between two points in a set. There exist numerous useful metrics for defining the distance between two points in Euclidean space, however, defining similar metrics for determining the distance between two locations of a finite rigid body is still an area of ongoing research, see Kazerounian and Rastegar, 1992, Bobrow and Park, 1995, Park, 1995, Martinez and Duffy, 1995, Larochele and McCarthy, 1995, Etzel and McCarthy, 1996, Gupta, 1997, Tse and Larochele, 2000, Chirikjian, 1998, and Belta and Kumar, 2002. In the cases of two locations of a finite rigid body in either  $SE(3)$  (spatial locations) or  $SE(2)$  (planar locations) any metric used to measure the distance between the locations yields a result which depends upon the chosen reference frames, see Bobrow and Park, 1995 and Martinez and Duffy, 1995. However, a metric that is independent of these choices, referred to as being bi-invariant, is desirable. Interestingly, for the specific case of orienting a finite rigid body in  $SO(n)$  bi-invariant metrics do exist. For example, Ravani and Roth, 1983 defined the distance between two orientations of a rigid body in space as the magnitude of the difference between the associated quaternions and a proof that this metric is bi-invariant may be found in Larochele and McCarthy, 1995.

Larochele and McCarthy, 1995 and Larochele, 1994 presented an algorithm for approximating displacements in  $SE(2)$  with spherical orientations in  $SO(3)$ . By utilizing the bi-invariant metric of Ravani and Roth, 1983 they arrived at an approximate bi-invariant metric for planar locations in which the error induced by the spherical approximation is of the order  $\frac{1}{R^2}$ , where  $R$  is the radius of the approximating sphere (hence the term *projective displacement metric*). Their algorithm for an approximately bi-invariant metric is based upon an algebraic formulation which utilizes Taylor series expansions of *sine()* and *cosine()* terms in homogeneous transforms, see McCarthy, 1983. Etzel and McCarthy, 1996 extended this work to spatial displacements by using orientations in  $SO(4)$  to approximate locations in  $SE(3)$ . Their algorithm is also based upon Taylor series expansions of *sine()* and *cosine()* terms, see Ge, 1994, and here too the error is of the order  $\frac{1}{R^2}$ .

This paper presents an efficient alternative approach for defining approximately bi-invariant projection metrics on  $SE(n)$  to those presented by Larochele and McCarthy, 1995 and Etzel and McCarthy, 1996. Here, the underlying geometrical motivations are the same- to approximate displacements with hyperspherical rotations. However, an alternative approach for reaching the same goal is presented. We utilize the sin-

gular value and polar decompositions to yield projections of planar and spatial finite displacements onto hyperspherical orientations.

## 2. Projecting SE(n) onto SO(n+1)

First, we review how spherical displacements may be used to approximate planar displacements with some finite error associated with the radius  $R$  of the sphere, see Larochelle, 1999 and Larochelle and McCarthy, 1995. This approach is based upon the work of McCarthy, 1983 in which he examined spherical and 3-spherical motions with instantaneous invariants approaching zero and showed that these motions may be identified with planar and spatial motions, respectively.

Recall that a general planar displacement  $(a, b, \alpha)$  in the  $z = R$  plane (an element of SE(2)) may be expressed as a homogeneous coordinate transformation (an element of H(2)),

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = [A_p] \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha & a \\ \sin \alpha & \cos \alpha & b \\ 0 & 0 & R \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \quad (1)$$

Now consider a general spherical displacement in which the parameters used to describe the displacement are the three angles longitude( $\theta$ ), latitude( $\phi$ ), and roll( $\psi$ ), see Fig. 1. Using these parameters a general spherical displacement may be written as,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = [A_s] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{Rot}(y, \theta) \text{Rot}(x, -\phi) \text{Rot}(z, \psi) \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (2)$$

We now define  $\hat{a} = R\theta$  as the longitudinal arc length and  $\hat{b} = R\phi$  as the latitudinal arc length. If we consider displacements in the  $z = R$  plane and expand the trigonometric functions *sine()* and *cosine()* using a Taylor series about 0 and substitute the angles  $\theta$  and  $\phi$  from above into the expansions then we may rewrite Eq. 2 as,

$$\begin{aligned} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &= \begin{bmatrix} \cos \psi & -\sin \psi & \hat{a} \\ \sin \psi & \cos \psi & \hat{b} \\ 0 & 0 & R \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\ &+ \frac{1}{R} \begin{bmatrix} 0 & 0 & 0 \\ -\hat{a} \cos \psi - \hat{b} \sin \psi & \hat{a} \sin \psi - \hat{b} \cos \psi & -\frac{1}{2}(\hat{a}^2 + \hat{b}^2) \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\ &+ O\left(\frac{1}{R^2}\right). \end{aligned} \quad (3)$$

Note that the first term of Eq. 3 is identical to Eq. 1 and we may approximate planar displacements  $(a, b, \psi)$  with some finite error that is

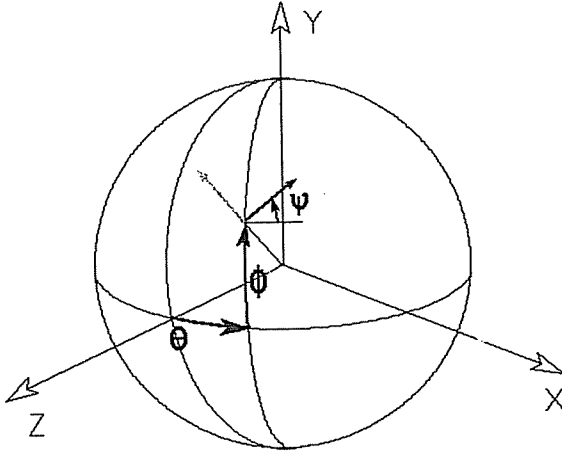


Figure 1. Planar Case:  $SE(2) \Rightarrow SO(3)$

associated with the radius of the sphere. From Eq. 3 we make the following identifications:  $\hat{a} \Rightarrow a$ ,  $\hat{b} \Rightarrow b$ , and,  $\psi \Rightarrow \alpha$ . Using the definition of the arc lengths and the radius of the sphere we obtain the three angles;  $\theta$ ,  $\phi$ , and  $\psi$ , which describe the spherical displacement on the sphere of radius  $R$  that approximates the prescribed planar displacement:  $\theta = \frac{a}{R}$ ,  $\phi = \frac{b}{R}$ , and,  $\psi = \alpha$ .

Etzel and McCarthy, 1996 extended the above methodology to spatial displacements by using orientations in  $SO(4)$  to approximate locations in  $SE(3)$ . They showed that a  $4 \times 4$  homogeneous transform representation of  $SE(3)$  can be approximated by a pure rotation  $[D]$  in  $SO(4)$ ,

$$[D] = [J(\alpha, \beta, \gamma)][K(\theta, \phi, \psi)] \quad (4)$$

where,

$$J(\alpha, \beta, \gamma) = \begin{bmatrix} \cos \alpha & 0 & 0 & \sin \alpha \\ -\sin \beta \sin \alpha & \cos \beta & 0 & \sin \beta \cos \alpha \\ -\sin \gamma \cos \beta \sin \alpha & -\sin \gamma \sin \beta & \cos \gamma & \sin \gamma \cos \beta \cos \alpha \\ -\cos \gamma \cos \beta \sin \alpha & -\sin \beta \cos \gamma & -\sin \gamma & \cos \gamma \cos \beta \cos \alpha \end{bmatrix}$$

and,

$$K(\theta, \phi, \psi) = \begin{bmatrix} & 0 \\ [A_s] & 0 \\ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The angles  $\alpha$ ,  $\beta$  and  $\gamma$  are defined as follows:  $\tan(\alpha) = \frac{d_x}{R}$ ,  $\tan(\beta) = \frac{d_y}{R}$ , and  $\tan(\gamma) = \frac{d_z}{R}$  where  $d_x$ ,  $d_y$ , and  $d_z$  are the components of the transla-

tion vector  $\mathbf{d}$  of the displacement and  $R$  is the radius of the hypersphere. A conceptual representation, analogous to Fig. 1, can be seen in Fig. 2.

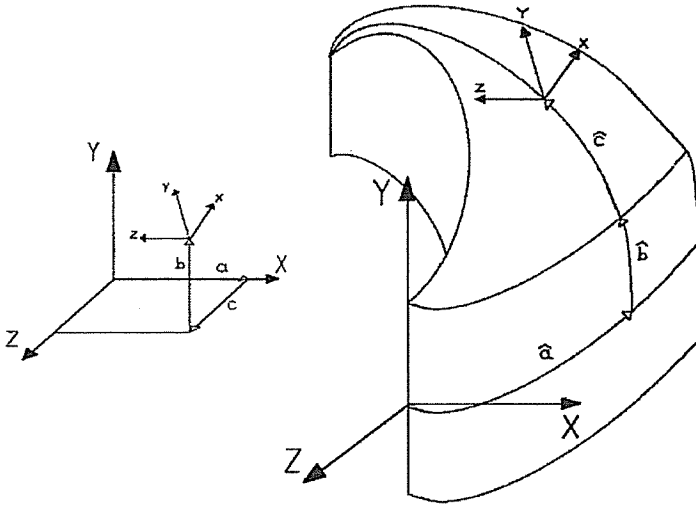


Figure 2. Spatial Case:  $SE(3) \Rightarrow SO(4)$  (figure from McCarthy, 1983)

### 3. The SVD Based Projection

This approach, analogous to the works reviewed above, also uses hyperdimensional rotations to approximate displacements. However, this new technique uses products derived from the singular value decomposition (SVD) of the homogeneous transform to realize the projection of  $SE(n-1)$  onto  $SO(n)$ . The general approach here is based upon preliminary works reported in Laroche and Dees, 2002 and Dees, 2001.

Consider the space of  $(n \times n)$  matrices as shown in Fig. 3. Let  $[T]$  be a  $(n \times n)$  homogeneous transform that represents an element of  $SE(n-1)$ . Note that  $[T]$  defines a point in  $R^{n^2}$ .  $[A]$  is the desired element of  $SO(n)$  nearest  $[T]$  when it lies in a direction orthogonal to the tangent plane of  $SO(n)$  at  $[A]$ .

The following theorem, based upon related works by Hanson and Norris, 1981 provides the foundation for the projection,

**Theorem 1** Given any  $(n \times n)$  matrix  $[T]$  the closest element of  $SO(n)$  is given by:  $[A] = [U][V]^T$  where  $[T] = [U][diag(s_1, s_2, \dots, s_n)][V]^T$  is the SVD of  $[T]$ .

Shoemaker and Duff, 1992 prove that matrix  $[A]$  satisfies the following optimization problem: Minimize:  $\|[A] - [T]\|_F^2$  subject to:  $[A]^T[A] - [I] =$

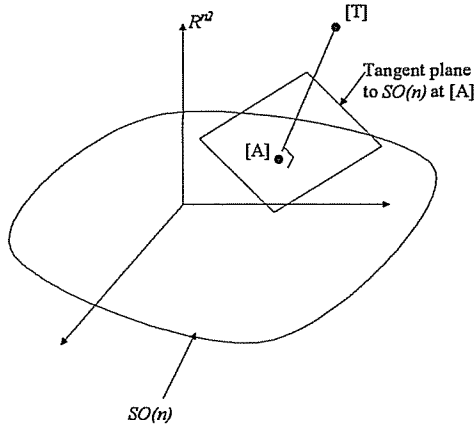


Figure 3. General Case:  $SE(n-1) \Rightarrow SO(n)$

$[0]$ , where  $\| [A] - [T] \|_F^2 = \sum_{i,j} (a_{ij} - t_{ij})^2$  is used to denote the Frobenius norm. Since  $[A]$  minimizes the Frobenius norm in  $R^{n^2}$  it is the element of  $SO(n)$  that lies in a direction orthogonal to the tangent plane of  $SO(n)$  at  $[R]$ . Hence,  $[A]$  is the closest element of  $SO(n)$  to  $[T]$ . Moreover, for full rank matrices the SVD is well defined and unique. We now restate Th. 1 with respect to the desired SVD based projection of  $SE(n-1)$  onto  $SO(n)$ ,

**Theorem 2** For  $[T] \in SE(n-1)$  and  $[T] = [U][diag(s_1, s_2, \dots, s_{n-1})][V]^T$  if  $[A] = [U][V]^T$  then  $[A]$  is the unique element of  $SO(n)$  nearest  $[T]$ .

Recall that  $[T]$ , the homogenous representation of  $SE(n)$ , is full rank (McCarthy, 1990) and therefore  $[A]$  exists, is well defined, and unique.

#### 4. The PD Based Projection

The polar decomposition, though perhaps less known than the SVD, is quite powerful and actually provides the foundation for the SVD. The polar decomposition theorem of Cauchy states that "a non-singular matrix equals an orthogonal matrix either pre or post multiplied by a positive definite symmetric matrix", see Halmos, 1958. With respect to our application, for  $[T] \in SE(n-1)$  its PD is  $[T] = [P][Q]$ , where  $[P]$  and  $[Q]$  are  $(n \times n)$  matrices such that  $[P]$  is orthogonal and  $[Q]$  is positive definite and symmetric. Recalling the properties of the SVD, the decomposition of  $[T][U][diag(s_1, s_2, \dots, s_{n-1})][V]^T$ , yields matrices  $[U]$  and  $[V]$  that are orthogonal and matrix  $[diag(s_1, s_2, \dots, s_{n-1})]$  which is positive definite and symmetric. Hence, for  $[A] = [U][V]^T$  we have  $[A]$

$= [P]$  and conclude that the polar decomposition yields the same element of  $SO(n)$ .

## 5. Computational Issues

Often, the evaluation of the singular value decomposition is implemented in code by computing the eigenvalues and eigenvectors of the matrix since the singular values are the positive square roots of the eigenvalues of  $[T][T]^T$  and the columns of  $[U]$  and  $[V]$  are the normed eigenvectors of  $[T][T]^T$  and  $[T]^T[T]$  respectively. However, we are computing the SVD of a homogeneous transform representing  $SE(n-1)$ . The eigenvalue and eigenvectors of  $SE(2)$  and  $SE(3)$  are well known and should be exploited to facilitate the computations, see McCarthy, 1990.

With regard to the PD, a simple and efficient iterative algorithm exists for its evaluation. Dubrulle, 1999 provides an algorithm that produces monotonic convergence in the Frobenius norm that "... generally delivers an IEEE double-precision solution in  $\sim 10$  or fewer steps". A MatLab implementation of Dubrulle's algorithm is shown in Fig. 4.

```
function P=polar(T)
%
% initialization
%
P=T;
limit = (1 + eps) * sqrt(size(T,2));
T = inv(P');
g = sqrt(norm(T,'fro')/norm(P,'fro'));
P=0.5*(g*P+(1/g)*T);
f = norm(P,'fro');
pf = inf;
%
% iteration
%
while (f>limit) & (f<pf)
    pf = f;
    T = inv(P');
    g=sqrt(norm(T,'fro')/f);
    P=0.5*(g*P+(1/g)*T);
    f=norm(P,'fro');
end
return
```

Figure 4. Dubrulle's PD Algorithm: MatLab Implmentation

Finally, it is important to recall that both the SVD and PD based projections of  $SE(n-1)$  onto  $SO(n)$  are coordinate frame and unit dependent. This is true for all metrics on spatial and planar displacements as no bi-invariant metric exists, see Bobrow and Park, 1995 and Martinez and Duffy, 1995. Note however that these mappings project  $SE(n-1)$  onto  $SO(n)$  and bi-invariant metrics do exist on  $SO(n)$ .

## 6. One metric on $SO(n)$

One useful and easily computed metric  $d$  on  $SO(n)$  follows. Given two elements  $[A_1]$  and  $[A_2]$  of  $SO(n)$  we can define a metric using the Frobenius norm as,

$$d = \|[I] - [A_2][A_1]^T\|_F. \quad (5)$$

It is straightforward to verify that this is a valid metric on  $SO(n)$ , see Schilling and Lee, 1988.

## 7. Case Study-1

Consider a planar displacement  $(a, b, \alpha) = (1, 1, 45)$ . Its corresponding element of  $SE(2)$  is  $[T]$  and we compute its projection  $[A]$  onto  $SO(3)$  using either technique presented here and yield:

$$[T] = \begin{bmatrix} 0.7071 & 0.7071 & 1 \\ 0.7071 & 0.7071 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

and

$$[A] = \begin{bmatrix} 0.5774 & 0.7071 & 0.4082 \\ 0.5774 & 0.70714 & 0.4082 \\ -0.5774 & 0.0 & 0.8165 \end{bmatrix}. \quad (7)$$

It is illustrative compute the angle and axis of rotation  $\eta = 56.60(\text{deg})$  and  $\mathbf{s} = [-0.2445 \ 0.5903 \ 0.7693]^T$ , see Fig. 1. Moreover, the longitude, latitude, and roll angles associated with  $[A]$  are:  $\theta = 25.56$ ,  $\phi = 24.09$ , and  $\psi = 39.23(\text{deg})$ . Finally, using the definitions of the longitudinal and latitudinal arc lengths  $R = 2.2674$  and from Eq. 5, we have  $\|[T]\| = 1.3409$ .

## 8. Case Study-2

Consider a spatial displacement  $(d_x, d_y, d_z, \theta, \phi, \psi) = (1, 2, 3, 10, 30, 75)$ . We proceed as above and yield the following:

$$[T] = \begin{bmatrix} 0.1710 & -0.9737 & 0.1540 & 1.0000 \\ 0.8365 & 0.2241 & 0.5000 & 2.0000 \\ -0.5206 & 0.0403 & 0.8529 & 3.0000 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

and

$$[A] = \begin{bmatrix} 0.1604 & -0.9584 & 0.0103 & 0.2357 \\ 0.8152 & 0.2547 & 0.2199 & 0.4714 \\ -0.5526 & 0.0861 & 0.4327 & 0.7071 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$



and from Eq. 5, we have  $\| [T] \| = 2.3155$ .

## 9. Conclusions

We have presented two new methods for approximate bi-invariant metrics on  $SE(n)$ . These methods are based on projections of  $SE(n)$  onto  $SO(n+1)$  that utilize the singular value and polar decompositions of the homogeneous transform representations of  $SE(n)$ . It was shown that both methods yield the same projection that determines the element of  $SO(n+1)$  nearest the given element of  $SE(n)$ . Any of the infinite bi-invariant metrics on  $SO(n+1)$  may then be used to measure the *distance* between any two spatial displacements  $SE(n)$ . The results are PD and SVD based projection techniques that yield two approximately bi-invariant metrics on  $SE(n)$ . These metrics have applications in motion synthesis, robot calibration, motion interpolation, and hybrid robot control.

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